The Iterative Logistic Map

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General form of the Iterative Logistic Map (ILM)

The logistic curve of Verhulst (population growth):

$$\frac{dx}{dt} = \lambda x(1-x)$$

A similar looking recurrence relation is the logistic map !

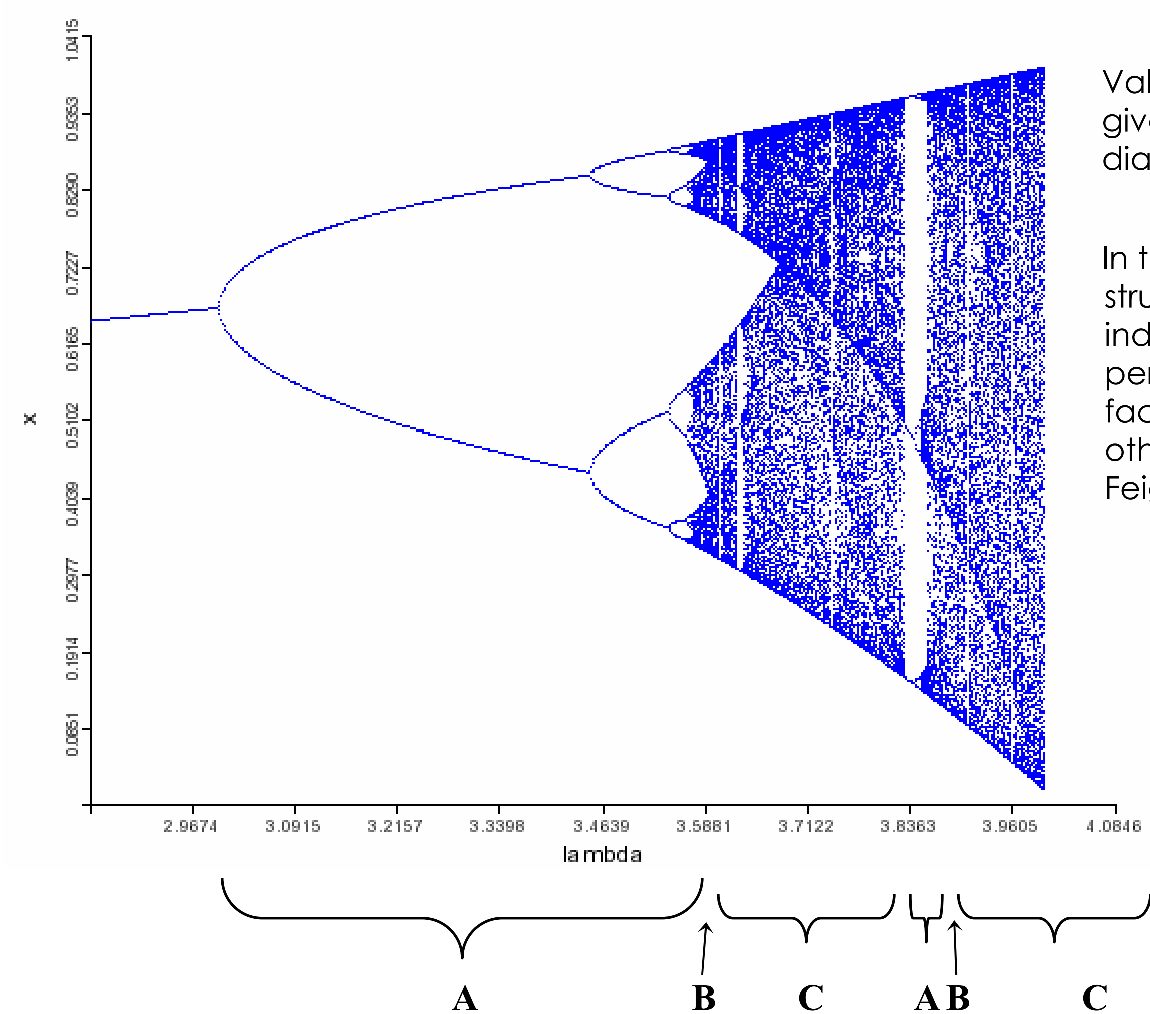
$$x_t = \lambda x_{t-1}(1-x_{t-1}) \qquad 0 < x_0 < 1 \,,\, 0 \le \lambda \le 4$$

Displays complicated behaviour very sensitive to λ and x_0 :

$\lambda < 0$	another bifurcating series
$0 \le \lambda \le 1$	series converges to 0
$1 < \lambda \le 3$	convergence to a value $0 \le x \le 2/3$ which depends on λ
$3 < \lambda \le 4$	oscillations with "period doubling" and chaotic regions
$\lambda > 4$	chaotic divergent region

Verhulst, P.-F. "Recherches mathématiques sur la loi d'accroissement de la population."
Nouv. mém. de l'Academie Royale des Sci. et Belles-Lettres de Bruxelles **18**, 1-41, 1845.

Period doubling and chaotic regions of the ILM



Values of x after convergence from gives the bifurcation, or Feigenbaum diagram.

In the period doubling regions the structures of higher periods are indistinguishable from the previous periods when magnified by specific factors which are constant even for other period doubling functions - Feigenbaum's constant(s)

δ = 4.669201609102990 …

α = -2.502907875 …

A = period doubling region
B = Accumulation point
C = chaotic region

But what does it do ?

- Period doubling is found in several naturally occuring chaotic phenomena, such as the beating of heart muscle and dripping taps.
- Provides a simple model for population growth:
    - At low breeding rates, $\lambda \leq 1$, the population eventually drops to zero
    - With intermediate breeding rates, $1 < \lambda \leq 3$, the population stabilises at a single value
    - At high breeding rates, $\lambda > 3$, there are population explosions followed by mass extinctions.
- The logistic map can be used to generate ‘random numbers’.

Some more chaotic systems and cellular automata

- Lorentz attractor (meteorology)
  - Iterative solution for systems of differential equations.
- Ising model (magnetization)
  - Monte carlo simulation of atom spin flipping at fixed lattice sites.
- Conway's game of life
  - A grid, two states and a few simple rules gives rise to a plethora of recurrent systems.
- 'BZ reaction' (reaction kinetics)
  - Redox reaction with autocatalysis
- Lattice Boltzmann (rheology)
  - Solution of Navier-Stokes equations for the flow of incompressible fluids.
- Disinfection simulation
  - Can the same principles be used to model variability in disinfection ?